RESEARCH PROPOSAL:

MATHEMATICAL STUDY OF COULOMB BRANCES OF GAUGE THEORIES

G : a compact Lie group say $U(N)$

$\mathbf{M}$: a quaternionic representation

say $\mathbf{M} = 0$, $\mathbb{H}^N$

$\qquad\qquad\qquad\qquad \mathbb{H}^N \cong \mathbb{C}^N \oplus (\mathbb{C}^N)^*$ $\qquad j(v_1, v_2^t) = (-\overline{v_2}, \overline{v_1}^t)$

$\underset{\text{physics}}{\rightsquigarrow}$ $N=4$ SUSY gauge theory $d=3$ $\underset{\text{physics}}{\rightsquigarrow}$ Coulomb branch $\mathcal{M}_C$

↑ not well-defined object

↑ a hyperKähler manifold with an $SU(2)$-action

possibly singular

Several properties

1. $\dim_{\mathbb{H}} \mathcal{M}_C = \operatorname{rk} G$ $\qquad$ in fact $\mathcal{M}_C \sim (\mathbb{R}^3 \times S^1)^{\operatorname{rk}} / W$ classically

2. $G = SU(2)$, $\mathbf{M} = 0$ $\Rightarrow \mathcal{M}_C = AH$

   $G = U(1)$, $\mathbf{M} = \mathbb{H}$ $\Rightarrow \mathcal{M}_C = TN$ $\qquad$ etc.

3. When $\mathcal{M}_C$ is nonsingular,

   gauge theory $\underset{\text{equiv.}}{\cong}$ topological σ-model with target $\mathcal{M}_C$

AIM define $\mathcal{M}_C$ in a rigorous way bypassing physics.

CURRENT STATUS

Yes, at least $\mathcal{M}_C$: holo. symp. variety (without Riem. metric)

with $\mathbf{M} = \mathbf{N} \oplus \mathbf{N}^*$

(with Braverman, Finkelberg)

$A := \mathbb{C}[\mathcal{M}_C]$ = ring of (algebraic) functions on $\mathcal{M}_C$

commutative ring

$\mathcal{M}_C$ is recovered from A as $\mathcal{M}_C = \operatorname{Spec} A$.

We construct A. ($\leftarrow$ this is called the chiral ring in physics literature)

<u>Physical fact.</u>

Chiral ring is generated by monopole operators

— gauge fields with pt singularities

ANOTHER AIM (MORE AMBITIOUS)

CONSTRUCT (3d) TQFT ASSOCIATED WITH GAUGE THEORIES

3d TQFT

M^3 : closed 3-mfd $\rightsquigarrow Z(M^3) \in \mathbb{C}$

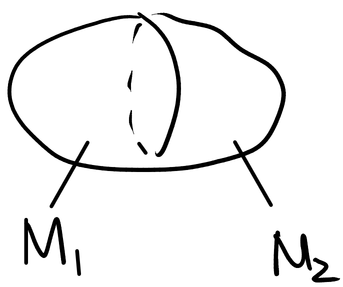
$\uparrow$ cpt no bdry

number

($\to$ corrected later)

$\Sigma^2 \rightsquigarrow Z(\Sigma^2)$: vector space

M^3 : mfd with bdry $\rightsquigarrow Z(M^3) \in Z(\partial M)$

$M =$ 

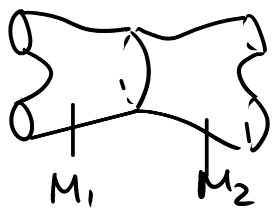
$\partial M_1 = - \partial M_2$

$\uparrow$ ori rev.

$Z(\partial M_1) = Z(\partial M_2)^*$

$Z(M) = \langle Z(M_1), Z(M_2) \rangle$

more generally



$Z(M) = Z(M_2) \circ Z(M_1)$
composition

Very roughly $Z(M^3) = \int_{A:\text{fields}} e^{S(A)} DA$

$\partial M^3 = \Sigma$ $\quad Z(M^3)(a) = \int_{A|_{\partial M^3} = a} e^{S(A)} DA$ $\quad \therefore Z(M^3) \in \text{Functional}\left(\begin{array}{l}\text{the space} \\ \text{of all} \\ \text{fields } a\end{array}\right)$

$\uparrow$ field on Σ

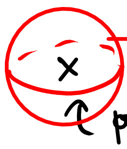
<u>Expectation</u>

The answer should depend on Riem. metric etc
in general, but thanks to SUSY,
it depends only on the topology of M in this case.

$Z(\Sigma) =$ "quantization of" $\text{Func}(\{a : \text{fields on } \Sigma\})$

Recall monopole operators :

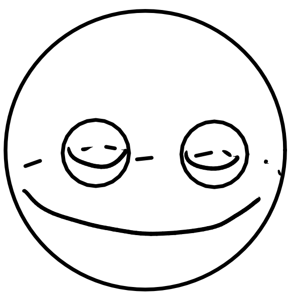
$\times$
$\uparrow$ pt sing. in M^3



$\uparrow$ pt sing. in M^3

$Z(\partial D^3) = Z(S^2)$

$\therefore$ monopole operator $\in Z(S^2)$



$Z(S^2)$ is equipped with
a commutative multiplication
from 3d TQFT.

$\therefore \boxed{A = \mathbb{C}[\mathcal{M}_C] \overset{\text{def.}}{=\!=} Z(S^2)}$!!

CONVENTIONAL APPROACHES TO TQFT

— consider PDE associated with $(G, \mathbb{M})$

$$\begin{cases} \not{D}_A \Phi = 0 \\ *F_A = \mu(\Phi) \end{cases}$$ on 3mfd M (and 2mfd Σ)

— consider its moduli space

$\mathcal{M}_M := \{\text{solutions}\} / \text{gauge equiv.}$

← expected to be 0-dimensional

$\mathcal{M}_\Sigma$ = the same for Σ ← positive dim. symplectic mfd

$b \uparrow$ ⌒ boundary value

$\mathcal{M}_M$ ← lagrangian

Then

- $Z(M) = \# \mathcal{M}_M$
- $Z(\Sigma) = H^*(\mathcal{M}_\Sigma)$
- $Z(M) = b_* [\mathcal{M}_M]$

This construction is partly justified

for $G = SO(3)$, $\mathbb{M} = 0$ Donaldson, Fukaya, ...

$G = U(1)$, $\mathbb{M} = \mathbb{H}$ Taubes, Kronheimer-Mrowka, Ozbath-Szabo, ...

TECHNICAL ISSUE : SINGULARITIES OF $\mathcal{M}_\Sigma$.

e.g. NO SATISFACTORY DEF. FOR $\mathcal{Z}(S^2) = H^*(\mathcal{M}_\Sigma)$

in fact $\dim \mathcal{Z}(S^2) = \mathcal{Z}(S^2 \times S^1) = -\frac{1}{12}$ (for Casson)

THEREFORE $\mathcal{Z}(\Sigma)$ SHOULD BE ∞-DIMENSIONAL.

— USE MORE ALGEBRAIC APPROACH